Announcements

1) Course Evaluations are online - do them!

2) HW # 5 up Thursday or Friday after the exam, due next Thursday (50 pts)

Recall: We were solving

$$r^2 f''(r) + r f'(r) + \alpha r^2 f(r) = 0.$$

We supposed

$$f(r) = \sum_{n=0}^{\infty} a_n r^n$$

and obtained

$$\boxed{a_n = 0, \ n \text{ odd}}$$ and

for $n = 2k$ even, $k = 1, 2, 3, \ldots$

$$\boxed{a_{2k} = \frac{\alpha^k a_0}{(k!)^2 4^k}}$$

Easy Case: $\alpha = 0$

$$a_{2k} = \frac{\alpha^k a_0}{(k!)^2 4^k} = 0$$

for all $k \geq 1$, we get $\boxed{f(r) = a_0}$, a constant.

But if $\alpha = 0$, then our original equation becomes

$$r^2 f''(r) + r f'(r) = 0$$

Cauchy-Euler

Solutions $f(r) = r^k$ for some constants k.

Substituting, we get

$$k(k-1) r^k + k r^k = 0$$

$$k^2 = 0, \quad k = 0$$

Then f is constant, which agrees with our series solution.

If $\alpha \neq 0$: For which values of r does the series converge?

Ratio Test

Given the power series

$$\sum_{n=0}^{\infty} \underbrace{\alpha_n (x-c)^n}_{b_n(x)}, \text{ consider}$$

$$\lim_{n\to\infty} \left| \frac{b_{n+1}(x)}{b_n(x)} \right|.$$

If

1) $\lim_{n \to \infty} \left| \frac{b_{n+1}(x)}{b_n(x)} \right| < 1$, then

the series converges.

2) $\lim_{n \to \infty} \left| \frac{b_{n+1}(x)}{b_n(x)} \right| > 1$, then

the series diverges

3) $\lim_{n \to \infty} \left| \frac{b_{n+1}(x)}{b_n(x)} \right| = 1$, you

know nothing.

Observations:

1) When running the test, you will either find the limit is less than one for all values of x or a number $L \geq 0$ such that the series converges when $|x-c| < L$. We call L the radius of convergence (in first case, $L = \infty$).

2) Problems whenever $b_n(x) = 0$ - can't divide! We can still (sometimes) use the test, but we'll have to be careful.

For our series $(a_0 \neq 0)$

$$b_n(x) = \begin{cases} \boxed{0, \quad n \text{ odd}} & \text{bad!} \\ \dfrac{\alpha^{n/2} a_0}{\left(\frac{n}{2}!\right)^2 2^n}, & n \text{ even} \end{cases}$$

Consider

$$a_0 + \sum_{k=1}^{\infty} \frac{\alpha^k a_0 x^{2k}}{(k!)^2 4^k},$$ apply

ratio test.

$$b_k(x) = \frac{a_0 \alpha^k x^{2k}}{(k!)^2 4^k}$$

$$b_{k+1}(x) = \frac{a_0 \alpha^{k+1} x^{2k+2}}{((k+1)!)^2 4^{k+1}}$$

$$\frac{b_{k+1}(x)}{b_k(x)} = \frac{\alpha x^2}{4} \cdot \left(\frac{k!}{(k+1)!}\right)^2$$

$$\frac{k!}{(k+1)!} = \frac{\cancel{k!}}{(k+1)\cancel{k!}} = \frac{1}{k+1}$$

So

$$\left|\frac{b_{k+1}(x)}{b_k(x)}\right| = \frac{|\alpha|}{4} x^2 \cdot \frac{1}{(k+1)^2}$$

$$\longrightarrow 0$$

for all values of x.

So regardless of α, the radius of convergence of our series is ∞ (converges for all real numbers).

Remark: If $\alpha = 1$,

we get

$$f(r) = a_0\left(1 + \sum_{k=1}^{\infty} \frac{r^{2k}}{(k!)^2 4^k}\right)$$

With $a_0 = 1$,

$f(r) = J_0(r)$, the

0^{th} Bessel Function

Differentiation and Integration

If $f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$

has radius of convergence $L > 0$, then for all x with $\boxed{|x-c| < L,}$

1) $f'(x) = \sum_{n=0}^{\infty} n a_n (x-c)^{n-1}$

2) $\int f(x)dx = D + \sum_{n=0}^{\infty} \frac{a_n (x-c)^{n+1}}{n+1}$